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LETTER TO THE EDITOR

There Is No Error in the Kleiser–Schumann Influence Matrix Method

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In 1980 Kleiser and Schumann [1] introduced the influence matrix method for solving the coupled velocity–pressure equations of the Stokes problem which arises when the incompressible Navier–Stokes equations are discretized in time. The influence matrix method is based on a suitable decomposition of the desired solution into a sum of solutions of subproblems which can be solved efficiently in a sequential manner. The technique guarantees exact fulfillment of the continuity equation by employing the proper boundary conditions for the pressure Poisson equation, which are obtained in the solution algorithm by enforcing the divergence-free condition on the boundary. While the original paper [1] treated the case of plane channel flow, a more general formulation of the influence matrix method was given in [2]. Numerous authors have since been using this method, or variants thereof, as well as its obvious extension to the analogous problem of the treatment of the vorticity boundary condition when the streamfunction-vorticity formulation of the basic equations is used.

In two recent papers [3, 4] the treatment of truncation errors in the original formulation of the influence matrix method was criticized. The claim was that the algorithm described in [1] contains some wrong boundary conditions which eventually lead to numerical solutions with significant divergence errors. It is the purpose of the present Note to clarify that the method given in [1] is indeed correct, and that the criticism in [3, 4] originates from a misunderstanding of the original algorithm. To make the point of misunderstanding clear, we will discuss the key points in the following. For a more detailed description of the influence-matrix technique, we refer the reader to the references cited and to the presentation given in Chapter 7.3.1. of the book by Canuto *et al.* [5].

Kleiser and Schumann [1] employed the influence matrix method within a spectral scheme developed for plane channel flow. A mixed explicit/implicit time discretization was utilized

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along with Fourier expansions for the spatial discretization in the wall-parallel directions. This discretization results in a set of four one-dimensional Helmholtz equations in the wall-normal direction which have to be solved for each Fourier mode at each time step. The first three of these equations are the discretized momentum equations, while the fourth is the discretized Poisson equation for the pressure P^{n+1} at the new time level t^{n+1}.

If no-slip conditions are prescribed at the walls, only the Poisson equation and the Helmholtz equation for the wall-normal velocity W^{n+1} are coupled. An efficient solution of the coupled equations for P^{n+1} and W^{n+1} poses difficulties, since natural boundary conditions are available for the velocity only (homogeneous Dirichlet conditions $W^{n+1} = 0$ are required at the no-slip walls). The proper pressure boundary conditions are not known *a priori*, but are determined implicitly by the solution which has to satisfy the homogeneous Neumann condition $DW^{n+1} = 0$ as a consequence of continuity (D represents the operator of the first derivative $\partial/\partial z$ in the wall-normal direction). This system of two coupled Helmholtz equations for P^{n+1} and W^{n+1}, together with homogeneous Dirichlet and Neumann boundary conditions for W^{n+1}, is commonly termed the "A-Problem" (see Canuto *et al.* [5, pp. 216–217]. It is given by Eq. (1) in [3].

For the solution of the A-Problem Kleiser and Schumann devised the influence-matrix technique which allows us to derive the correct boundary conditions for the pressure P^{n+1} directly from the continuity condition. In this method a so-called "B-Problem" is solved rather than the original A-Problem which differs from the latter by the fact that the Neumann conditions for W^{n+1} are replaced by some Dirichlet conditions for P^{n+1}. The space of all possible solutions of the coupled system (P^{n+1}, W^{n+1}) with arbitrary Dirichlet conditions for P^{n+1} is spanned by a particular solution of this system and two solutions of the associated homogeneous problem (Eqs. (2)–(4) in [3]). These three solutions will be referred to as partial solutions in the remainder. Within the space of solutions the desired one is uniquely determined by the requirement $DW^{n+1} = 0$ at the walls.

The individual Helmholtz equations that arise in the method outlined above have to be solved numerically which implies some finite truncation errors. These approximation errors deserve special care if the final solution is to be divergence-free to within machine accuracy. This holds for any discretization method employed in the numerical solution of the Helmholtz equations, whether it is a Chebyshev tau method (as in [1]), a Chebyshev collocation technique (see [5]) or any other finite numerical scheme. According to [1], the respective approximation errors can be taken into account in such a way that the continuity equation is fulfilled exactly. This involves further auxiliary Helmholtz equations which differ from the original ones with respect to the right-hand sides and the boundary conditions. These auxiliary equations may be introduced either on the level of the A-Problem or on the level of the B-Problem. The latter approach was adopted by Kleiser and Schumann, who applied the correction algorithm to each of the three partial solutions of the B-Problem. In this case the required auxiliary solution ($\tilde{P}$, $\tilde{W}$) has to satisfy homogeneous Dirichlet conditions for both $\tilde{W}$ and $\tilde{P}$. Although the correction must be conducted for each of the partial problems separately, the involved auxiliary solution is the same for the three cases.

In the alternative approach, which is described in [3], the correction is conducted on the level of the A-Problem rather than on the level of the B-Problem. This requires the solution of an auxiliary A-Problem with the boundary conditions $\tilde{W} = D\tilde{W} = 0$ (Eqs. (14.a) and (14.b) in [3]). For the solution of this A-Problem again an influence-matrix technique may be employed. The final solution is then obtained from a linear combination of the solutions of the two A-Problems.

In principle, the correction algorithms employed in the two approaches outlined above are similar. It must be stressed, however, that they differ significantly in detail as becomes evident from the difference in the boundary conditions for the auxiliary solutions. Now the point of misunderstanding in [3] is that the author assumed that the auxiliary solutions of the B-Problem given in [1] were to be employed for a correction on the level of the A-Problem. If this is done, the correction algorithm will necessarily fail and the resulting flow fields will exhibit about the same divergence errors as the uncorrected solution. This happened in [3] and led to the false conjecture that Kleiser and Schumann [1] had given the wrong boundary conditions for the auxiliary solution. The algorithm which is then put forth as a corrected Kleiser–Schumann method is a valid method to deal with the approximation errors on the level of the A-Problem; it is, however, not a corrected version of the method presented in [1].

Kleiser and Schumann developed their method "in order to obtain an exactly divergence-free solution," as stated on page 169 of [1], and this was indeed achieved. The application of the influence-matrix technique leads to numerical solutions where div $\underline{u} = 0$ is fulfilled to machine precision (except for the loss of a few digits, in practice to about 13–14 rather than 16 decimal places on a CRAY Y-MP computer).

In conclusion, we wish to point out that the original algorithm of the influence matrix technique as described in [1] works perfectly and does not need any correction. It represents a very accurate and efficient method which has been applied successfully by many researchers to a variety of laminar, tansitional and turbulent flows in the past.

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